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Solution of Abel's Integral Equation Using Sadik Transform

Abstract

Abel's integral equation is an important singular integral equation and generally appears in many branches of sciences such as mechanics, atomic scattering, physics, electron emission, radio astronomy, X-ray radiography and seismology. In this paper, we use Sadik transform to solve Abel's integral equation and some numerical applications in application section are given to demonstrate the effectiveness of Sadik transform for solving Abel's integral equation. It is pointed out that Sadik transform give the exact solution of Abel's integral equation without any tedious calculation work.

Keywords: Abel's Integral Equation, Sadik Transforms Method, Inverse Sadik Transform, Convolution Theorem.

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Introduction:

In mathematical form, Abel's integral equation is given by [11, 13, 24, 29, 43-44]

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\, dt \quad (1)$$

where the functions $f(x)$ and $u(x)$ are known and unknown functions respectively.

In equation (1), the kernel of integral equation, $K(x,t) = \frac{1}{\sqrt{x-t}}$ becomes ∞ at $t = x$ so equation (1) is a singular integral equation.

Review of Literature:

Integral transforms are widely used mathematical techniques for solving advanced problems of science and engineering which mathematically express in terms of differential equations, delay differential equations, system of differential equations, partial differential equations, integral equations, system of integral equations, partial integro-differential equations and integro differential equations. Many researchers used different integral transforms (Laplace transform [1-2], Fourier transform [2], Mahgoub transform [3-11, 46-48], Kamal transform [12-19, 49], Aboodh transform [20-25, 50-55], Mohand transform [26-36, 45, 56-57], Elzaki transform [37-40, 58-60], Shehu transform [41-43, 61], Sumudu transform [44, 62-63] and Sadik transform [64-66]) for solving many problems of science and engineering.

Sadik transform of the function $F(t), t \geq 0$ was proposed by Sadikali in 2018 [64] as:

$$S\{F(t)\} = \frac{1}{v^{\beta}} \int_0^{\infty} F(t) e^{-tv^{\alpha}} dt = T(v^{\alpha}, \beta), \quad (2)$$

where v is complex variable and $\alpha \neq 0$ & β are any real numbers.

Here S is called the Sadik transform operator.

The Sadik transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. The mention two conditions are the only sufficient conditions for the existence of Sadik transforms of the function $F(t)$.

Aim of the Study:

In this paper, we are finding the solution of Abel's integral equation using Sadik transform and explain the complete procedure by giving some numerical applications in application section.

Some Useful Properties of Sadik Transform:

Linearity property of Sadik transforms:

If $S\{F_1(t)\} = T_1(v^{\alpha}, \beta)$ and $S\{F_2(t)\} = T_2(v^{\alpha}, \beta)$ then

$S\{aF_1(t) + bF_2(t)\} = aS\{F_1(t)\} + bS\{F_2(t)\}$

$\Rightarrow S\{aF_1(t) + bF_2(t)\} = aS\{F_1(t)\} + bS\{F_2(t)\} = aT_1(v^{\alpha}, \beta) + bT_2(v^{\alpha}, \beta).$

Proof By the definition of Sadik transform, we have

$$S\{F(t)\} = \frac{1}{v^\beta}\int_0^\infty F(t)e^{-tv^\alpha}dt$$

$$\Rightarrow S\{aF_1(t)+bF_2(t)\} = \frac{1}{v^\beta}\int_0^\infty [aF_1(t)+bF_2(t)]e^{-tv^\alpha}dt$$

$$\Rightarrow S\{aF_1(t)+bF_2(t)\}$$
$$= a.\frac{1}{v^\beta}\int_0^\infty F_1(t)e^{-tv^\alpha}dt + b.\frac{1}{v^\beta}\int_0^\infty F_2(t)e^{-tv^\alpha}dt$$

$$\Rightarrow S\{aF_1(t)+bF_2(t)\} = aS\{F_1(t)\}+bS\{F_2(t)\}$$

$$\Rightarrow S\{aF_1(t)+bF_2(t)\} = aT_1(v^\alpha,\beta)+bT_2(v^\alpha,\beta),$$

where a, b are arbitrary constants.

Change of scale property of Sadik transforms:

If Sadik transform of function $F(t)$ is $T(v^\alpha,\beta)$ then Sadik transform of function $F(at)$ is given by $\frac{1}{a}T\left(\frac{v^\alpha}{a},\beta\right)$.

Proof: By the definition of Sadik transform, we have

$$S\{F(at)\} = \frac{1}{v^\beta}\int_0^\infty F(at)e^{-tv^\alpha}dt$$

Put $at = p \Rightarrow adt = dp$ in above equation, we have

$$S\{F(at)\} = \frac{1}{a}.\frac{1}{v^\beta}\int_0^\infty F(p)e^{-\frac{pv^\alpha}{a}}dp$$

$$\Rightarrow S\{F(at)\} = \frac{1}{a}\left[\frac{1}{v^\beta}\int_0^\infty F(p)e^{-p\left(\frac{v^\alpha}{a}\right)}dp\right]$$

$$\Rightarrow S\{F(at)\} = \frac{1}{a}T\left(\frac{v^\alpha}{a},\beta\right)$$

Shifting property of Sadik transform:

If Sadik transform of function $F(t)$ is $T(v^\alpha,\beta)$ then Sadik transform of function $e^{at}F(t)$ is given by $T(v^\alpha - a,\beta)$.

Proof: By the definition of Sadik transform, we have

$$S\{e^{at}F(t)\} = \frac{1}{v^\beta}\int_0^\infty e^{at}F(at)e^{-tv^\alpha}dt$$

$$\Rightarrow S\{e^{at}F(t)\} = \frac{1}{v^\beta}\int_0^\infty F(t)e^{-(v^\alpha-a)t}dt$$

$$\Rightarrow S\{e^{at}F(t)\} = T(v^\alpha - a,\beta)$$

Sadik transform of the derivatives of the function $\mathbf{F(t)}$ [64, 65]:

If $S\{F(t)\} = T(v^\alpha,\beta)$ then

a) $S\{F'(t)\} = v^\alpha T(v^\alpha,\beta) - \frac{F(0)}{v^\beta}$

b) $S\{F''(t)\} = v^{2\alpha}T(v^\alpha,\beta) - \frac{F'(0)}{v^\beta} - v^\alpha\frac{F(0)}{v^\beta}$

c) $S\{F^{(n)}(t)\} = v^{n\alpha}T(v^\alpha,\beta) - v^{(n-1)\alpha}\frac{F(0)}{v^\beta} - v^{(n-2)\alpha}\frac{F'(0)}{v^\beta} - \cdots \ldots - \frac{F^{(n-1)}(0)}{v^\beta}$.

Convolution of two functions [2]:

Convolution of two functions $F_1(t)$ and $F_2(t)$ is denoted by $F_1(t)*F_2(t)$ and it is defined by

$$F_1(t)*F_2(t) = \int_0^t F_1(t-x)\,F_2(x)dx$$
$$= \int_0^t F_1(x)\,F_2(t-x)dx.$$

Convolution theorem for Sadik transforms:

If Sadik transform of functions $F_1(t)$ and $F_2(t)$ are $T_1(v^\alpha,\beta)$ and $T_2(v^\alpha,\beta)$ respectively then Sadik transform of their convolution $F_1(t)*F_2(t)$ is given by

$$S\{F_1(t)*F_2(t)\} = v^\beta S\{F_1(t)\}S\{F_2(t)\}$$
$$\Rightarrow S\{F_1(t)*F_2(t)\} = v^\beta\, T_1(v^\alpha,\beta)T_2(v^\alpha,\beta),$$

Proof: By the definition of Sadik transform, we have

$$S\{F_1(t)*F_2(t)\} = \frac{1}{v^\beta}\int_0^\infty [F_1(t)*F_2(t)]e^{-tv^\alpha}dt$$

$$\Rightarrow S\{F_1(t)*F_2(t)\} = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\left[\int_0^t F_1(t-x)\,F_2(x)dx\right]dt$$

By changing the order of integration, we have

$$S\{F_1(t)*F_2(t)\} = \int_0^\infty F_2(x)\left[\frac{1}{v^\beta}\int_x^\infty e^{-tv^\alpha}F_1(t-x)\,dt\right]dx$$

Put $t - x = p$ so that $dt = dp$ in above equation, we have

$$S\{F_1(t)*F_2(t)\} = \int_0^\infty F_2(x)\left[\frac{1}{v^\beta}\int_0^\infty e^{-(p+x)v^\alpha}F_1(p)\,dp\right]dx$$

$$\Rightarrow S\{F_1(t)*F_2(t)\}$$
$$= \int_0^\infty F_2(x)e^{-xv^\alpha}\left[\frac{1}{v^\beta}\int_0^\infty e^{-pv^\alpha}F_1(p)\,dp\right]dx$$
$$= \int_0^\infty F_2(x)e^{-xv^\alpha}\,[S\{F_1(t)\}]dx$$

$$\Rightarrow S\{F_1(t)*F_2(t)\} = [S\{F_1(t)\}]\int_0^\infty F_2(x)e^{-xv^\alpha}\,dx$$

$$\Rightarrow S\{F_1(t)*F_2(t)\} = [\,T_1(v^\alpha,\beta)]v^\beta\left[\frac{1}{v^\beta}\int_0^\infty F_2(x)e^{-xv^\alpha}\,dx\right]$$

$$\Rightarrow S\{F_1(t)*F_2(t)\} = v^\beta S\{F_1(t)\}S\{F_2(t)\}$$
$$\Rightarrow S\{F_1(t)*F_2(t)\} = v^\beta\, T_1(v^\alpha,\beta)T_2(v^\alpha,\beta).$$

Sadik Transform of Frequently Encountered Functions [64, 66]:

Table: 1

S.N.	$F(t)$	$S\{F(t)\} = T(v^\alpha,\beta)$
1.	1	$\frac{1}{v^{\alpha+\beta}}$
2.	t	$\frac{1}{v^{2\alpha+\beta}}$
3.	t^2	$\frac{2!}{v^{3\alpha+\beta}}$
4.	$t^n, n \in N$	$\frac{n!}{v^{(n+1)\alpha+\beta}}$
5.	$t^n, n > -1$	$\frac{\Gamma(n+1)}{v^{(n+1)\alpha+\beta}}$
6.	e^{at}	$\frac{1}{v^\beta(v^\alpha - a)}$
7.	sinat	$\frac{a}{v^\beta(v^{2\alpha}+a^2)}$
8.	cosat	$\frac{v^\alpha}{v^\beta(v^{2\alpha}+a^2)}$
9.	sinhat	$\frac{a}{v^\beta(v^{2\alpha}-a^2)}$

10.	$\cosh at$	$\frac{v^{\alpha}}{v^{\beta}(v^{2\alpha}-a^2)}$

Inverse Sadik Transform [66]:

If Sadik transform of $F(t)$ is $T(v^{\alpha},\beta)$ i.e. $S\{F(t)\}=T(v^{\alpha},\beta)$ then $F(t)$ is called the inverse Sadik transform of $T(v^{\alpha},\beta)$ and mathematical terms it can be expressed as $F(t)=S^{-1}\{T(v^{\alpha},\beta)\}$.

Here the operator S^{-1} is known as the inverse Sadik transform operator.

Linearity Property of Inverse Sadik Transforms:

If $S^{-1}\{T_1(v^{\alpha},\beta)\}=F_1(t)$ and $S^{-1}\{T_2(v^{\alpha},\beta)\}=F_2(t)$ then $S^{-1}\{aT_1(v^{\alpha},\beta)+bT_2(v^{\alpha},\beta)\}$

$=aS^{-1}\{T_1(v^{\alpha},\beta)\}+bS^{-1}\{T_2(v^{\alpha},\beta)\}$

$\Rightarrow S^{-1}\{aT_1(v^{\alpha},\beta)+bT_2(v^{\alpha},\beta)\}=a\,F_1(t)+b\,F_2(t)$, where a,b are arbitrary constants.

Inverse Sadik Transform of Frequently Encountered Functions [66]:

Table: 2

S.N.	$T(v^{\alpha},\beta)$	$F(t)=S^{-1}\{T(v)\}$
1.	$\frac{1}{v^{\alpha+\beta}}$	1
2.	$\frac{1}{v^{2\alpha+\beta}}$	t
3.	$\frac{1}{v^{3\alpha+\beta}}$	$\frac{t^2}{2!}$
4.	$\frac{1}{v^{(n+1)\alpha+\beta}}$	$\frac{t^n}{n!}$
5.	$\frac{1}{v^{(n+1)\alpha+\beta}}$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{1}{v^{\beta}(v^{\alpha}-a)}$	e^{at}
7.	$\frac{1}{v^{\beta}(v^{2\alpha}+a^2)}$	$\frac{\sin at}{a}$
8.	$\frac{v^{\alpha}}{v^{\beta}(v^{2\alpha}+a^2)}$	$\cos at$
9.	$\frac{1}{v^{\beta}(v^{2\alpha}-a^2)}$	$\frac{\sinh at}{a}$
10.	$\frac{v^{\alpha}}{v^{\beta}(v^{2\alpha}-a^2)}$	$\cosh at$

Sadik Transform Method for Solving Abel's Integral Equation

In this section, we present Sadik transform for the solution of Abel's integral equation.

Taking Sadik transform of both sides of equation (1), we have

$$S\{f(x)\}=S\left\{\int_0^x \frac{1}{\sqrt{x-t}}u(t)\,dt\right\}$$

$$\Rightarrow S\{f(x)\}=S\{x^{-1/2}*u(x)\} \quad (3)$$

Applying convolution theorem of Sadik transform in equation (3), we have

$$S\{f(x)\}=v^{\beta}S\{x^{-1/2}\}S\{u(x)\}$$

$$\Rightarrow S\{f(x)\}=v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right)S\{u(x)\}$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\frac{\alpha}{2}}}{\sqrt{\pi}}S\{f(x)\}$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\alpha}}{\pi}\left[v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right)S\{f(x)\}\right]$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\alpha}}{\pi}\left[v^{\beta}S\{x^{-1/2}\}S\{f(x)\}\right]$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\alpha}}{\pi}S\{x^{-1/2}*f(x)\}$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\alpha}}{\pi}\left[S\left\{\int_0^x \frac{1}{\sqrt{x-t}}f(t)\,dt\right\}\right]$$

$$\Rightarrow S\{u(x)\}=\frac{v^{\alpha}}{\pi}S\{F(x)\} \quad (4)$$

where $F(x)=\int_0^x \frac{1}{\sqrt{x-t}}f(t)\,dt$ (5)

Now applying the property, Sadik transform of derivative of the function, on equation (5), we have

$$S\{F'(x)\}=v^{\alpha}S\{F(x)\}-\frac{F(0)}{v^{\beta}}$$

$$\Rightarrow S\{F'(x)\}=v^{\alpha}S\{F(x)\}$$

$$\Rightarrow S\{F(x)\}=\frac{1}{v^{\alpha}}S\{F'(x)\} \quad (6)$$

Now from (4) and (6), we have

$$S\{u(x)\}=\frac{1}{\pi}S\{F'(x)\} \quad (7)$$

Applying inverse Sadik transform on both sides of equation (7), we get

$$u(x)=\frac{1}{\pi}F'(x)=\frac{1}{\pi}\frac{d}{dx}F(x) \quad (8)$$

Using (5) in (8), we have

$$u(x)=\frac{1}{\pi}\left[\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-t}}f(t)\,dt\right] \quad (9)$$

which is the required solution of (1).

Applications:

In this section, we present some numerical applications to demonstrate the effectiveness of Sadik transform for solving Abel's integral equation**.**

Application:1 Consider the Abel's integral equation:

$$x=\int_0^x \frac{1}{\sqrt{x-t}}u(t)\,dt \quad (10)$$

Taking Sadik transform of both sides of equation (10), we have

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$$S\{x\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \frac{1}{v^{2\alpha+\beta}} = S\{x^{-1/2} * u(x)\} \quad (11)$$

Applying convolution theorem of Sadik transform in equation (11), we have

$$\frac{1}{v^{2\alpha+\beta}} = v^{\beta} S\{x^{-1/2}\} S\{u(x)\}$$

$$\Rightarrow \frac{1}{v^{2\alpha+\beta}} = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{u(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{1}{\left[v^{\frac{3\alpha}{2}+\beta}\right]\sqrt{\pi}} \quad (12)$$

Applying inverse Sadik transform on both sides of equation (12), we get

$$u(x) = \frac{1}{\sqrt{\pi}} S^{-1}\left\{\frac{1}{\left[v^{\frac{3\alpha}{2}+\beta}\right]}\right\}$$

$$\Rightarrow u(x) = \frac{2}{\pi} x^{1/2} \quad (13)$$

which is the required solution of equation (10).

Application:2 Consider the Abel's integral equation:

$$1 + x + x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt \quad (14)$$

Taking Sadik transform of both sides of equation (14), we have

$$S\{1\} + S\{x\} + S\{x^2\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \frac{1}{v^{\alpha+\beta}} + \frac{1}{v^{2\alpha+\beta}} + \frac{2}{v^{3\alpha+\beta}} = S\{x^{-1/2} * u(x)\} \quad (15)$$

Applying convolution theorem of Sadik transform in equation (15), we have

$$\frac{1}{v^{\alpha+\beta}} + \frac{1}{v^{2\alpha+\beta}} + \frac{2}{v^{3\alpha+\beta}} = v^{\beta} S\{x^{-1/2}\} S\{u(x)\}$$

$$\Rightarrow \frac{1}{v^{\alpha+\beta}} + \frac{1}{v^{2\alpha+\beta}} + \frac{2}{v^{3\alpha+\beta}} = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{u(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{1}{\sqrt{\pi}}\left[\frac{1}{v^{\frac{\alpha}{2}+\beta}} + \frac{1}{v^{\frac{3\alpha}{2}+\beta}} + \frac{2}{v^{\frac{5\alpha}{2}+\beta}}\right] \quad (16)$$

Applying inverse Sadik transform on both sides of equation (16), we get

$$u(x) = \frac{1}{\sqrt{\pi}} S^{-1}\left\{\frac{1}{v^{\frac{\alpha}{2}+\beta}} + \frac{1}{v^{\frac{3\alpha}{2}+\beta}} + \frac{2}{v^{\frac{5\alpha}{2}+\beta}}\right\}$$

$$\Rightarrow u(x) = \frac{1}{\sqrt{\pi}}\left[S^{-1}\left\{\frac{1}{v^{\frac{\alpha}{2}+\beta}}\right\} + S^{-1}\left\{\frac{1}{v^{\frac{3\alpha}{2}+\beta}}\right\} + 2S^{-1}\left\{\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right\}\right]$$

$$\Rightarrow u(x) = \frac{1}{\pi}\left[x^{-1/2} + 2x^{1/2} + \frac{8}{3}x^{3/2}\right] \quad (17)$$

which is the required solution of equation (14).

Application:3 Consider the Abel's integral equation:

$$3x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt \quad (18)$$

Taking Sadik transform of both sides of equation (18), we have

$$3S\{x^2\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \frac{6}{v^{3\alpha+\beta}} = S\{x^{-1/2} * u(x)\} \quad (19)$$

Applying convolution theorem of Sadik transform in equation (19), we have

$$\frac{6}{v^{3\alpha+\beta}} = v^{\beta} S\{x^{-1/2}\} S\{u(x)\}$$

$$\Rightarrow \frac{6}{v^{3\alpha+\beta}} = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{u(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{6}{\sqrt{\pi}}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] \quad (20)$$

Applying inverse Sadik transform on both sides of equation (20), we get

$$u(x) = \frac{6}{\sqrt{\pi}} S^{-1}\left\{\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right\}$$

$$\Rightarrow u(x) = \frac{8}{\pi} x^{3/2} \quad (21)$$

which is the required solution of equation (18).

Application:4 Consider the Abel's integral equation:

$$\frac{4}{3} x^{3/2} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt \quad (22)$$

Taking Sadik transform of both sides of equation (22), we have

$$\frac{4}{3} S\{x^{3/2}\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = S\{x^{-1/2} * u(x)\} \quad (23)$$

Applying convolution theorem of Sadik transform in equation (23), we have

$$\sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = v^{\beta} S\{x^{-1/2}\} S\{u(x)\}$$

$$\Rightarrow \sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{\mathrm{u}(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{1}{v^{2\alpha+\beta}} \quad (24)$$

Applying inverse Sadik transform on both sides of equation (24), we get

$$u(x) = S^{-1}\left\{\frac{1}{v^{2\alpha+\beta}}\right\}$$

$$\Rightarrow u(x) = x \quad (25)$$

which is the required solution of equation (22).

Application:5 Consider the Abel's integral equation:

$$2\sqrt{x} + \frac{8}{3} x^{3/2} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt \quad (26)$$

Taking Sadik transform of both sides of equation (26), we have

$$2S\{x^{1/2}\} + \frac{8}{3} S\{x^{3/2}\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \sqrt{\pi}\left[\frac{1}{v^{\frac{3\alpha}{2}+\beta}}\right] + 2\sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = S\{x^{-1/2} * u(x)\} \quad (27)$$

Applying convolution theorem of Sadik transform in equation (27), we have

$$\sqrt{\pi}\left[\frac{1}{v^{\frac{3\alpha}{2}+\beta}}\right] + 2\sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = v^{\beta} S\left\{x^{-\frac{1}{2}}\right\} S\{u(x)\}$$

$$\Rightarrow \sqrt{\pi}\left[\frac{1}{v^{\frac{3\alpha}{2}+\beta}}\right] + 2\sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{u(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{1}{v^{\alpha+\beta}} + \frac{2}{v^{2\alpha+\beta}} \quad (28)$$

Applying inverse Sadik transform on both sides of equation (28), we get

$$u(x) = S^{-1}\left\{\frac{1}{v^{\alpha+\beta}} + \frac{2}{v^{2\alpha+\beta}}\right\}$$

$$= S^{-1}\left\{\frac{1}{v^{\alpha+\beta}}\right\} + 2S^{-1}\left\{\frac{1}{v^{2\alpha+\beta}}\right\}$$

$$\Rightarrow u(x) = 1 + 2x \quad (29)$$

which is the required solution of equation (26).

Application:6 Consider the Abel's integral equation:

$$\frac{3}{8}\pi x^2 = \int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt \quad (30)$$

Taking Sadik transform of both sides of equation (30), we have

$$\frac{3}{8}\pi S\{x^2\} = S\left\{\int_0^x \frac{1}{\sqrt{x-t}} u(t)\,dt\right\}$$

$$\Rightarrow \frac{3}{4}\pi\left(\frac{1}{v^{3\alpha+\beta}}\right) = S\{x^{-1/2} * u(x)\} \quad (31)$$

Applying convolution theorem of Sadik transform in equation (31), we have

$$\frac{3}{4}\pi\left(\frac{1}{v^{3\alpha+\beta}}\right) = v^{\beta} S\left\{x^{-\frac{1}{2}}\right\} S\{u(x)\}$$

$$\Rightarrow \frac{3}{4}\pi\left(\frac{1}{v^{3\alpha+\beta}}\right) = v^{\beta}\left(\frac{\sqrt{\pi}}{v^{\frac{\alpha}{2}+\beta}}\right) S\{u(x)\}$$

$$\Rightarrow S\{u(x)\} = \frac{3}{4}\sqrt{\pi}\left[\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right] \quad (32)$$

Applying inverse Sadik transform on both sides of equation (32), we get

$$u(x) = \frac{3}{4}\sqrt{\pi} S^{-1}\left\{\frac{1}{v^{\frac{5\alpha}{2}+\beta}}\right\}$$

$$\Rightarrow u(x) = x^{3/2} \quad (33)$$

which is the required solution of equation (30).

Conclusion:

In this paper, we have successfully discussed Sadik transform for the solution of Abel's integral equation. The given numerical applications in the application section explain the complete procedure for the solution of Abel's integral equation using Sadik transform. The results show Sadik transform is a powerful integral transform for the solution of Abel's integral equation. In the future, Sadik transform can be applied for solving other singular integral equations.

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